PHYSICAL REVIEW E **79**, 045101(R) (2009)

Synchronization with on-off coupling: Role of time scales in network dynamics

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(Received 27 November 2008; published 14 April 2009)

We consider the problem of synchronizing a general complex network by means of the on-off coupling strategy; in this case, the on-off time scale is varied from a very small to a very large value. In particular, we find that when the time scale is comparable to that of node dynamics, synchronization can also be achieved and greatly optimized for the upper bound of the stability region which nearly disappears, and the synchronization speed is accelerated a lot, independent of network topologies. Our study indicates that the time scale for network variation is of crucial importance for network dynamics and synchronization under the comparable time scale which is much more advantageous over other time scales. Both analysis and experiments confirm the conclusions.

DOI: 10.1103/PhysRevE.79.045101 PACS number(s): 89.75.Hc, 05.45.Xt, 87.23.Cc

Network-based approaches have attracted a lot of interests and have proven to be prominent candidates to describe sophisticated collaborative dynamics in many fields [1,2]. So far, the dynamics of complex network has been extensively investigated, with special emphasis on network synchronization and the interplay between the complexity in the overall topology and the local properties of the coupled oscillators [3,4].

However, the majority of works in this regard has focused on the synchronization of static network, whose coupling configuration in the network is time invariant, contrasting to the reality such as in biological, epidemiological, and social networks that the topologies can also evolve with agent dynamics as time goes by. Only recently, the case of time-varying networks has been taken into account [5–10], among which most of the researches are prone to the fast switching case; that is, the time scale of the variation in networks is much shorter than that of the oscillator dynamics. However, systems under different time scales of network variation may exhibit very different synchronous behaviors, i.e., the role of time scales for network synchronization could be of crucial importance. In particular, when the time scale is comparable to that of node dynamics, this situation, indeed, can be considered as a good model, such as for synaptic plasticity in neuronal networks [11], social or financial market adaptation dynamics [12], or mutation processes in biological systems [13], whose network evolution takes place over characteristic time scales that commensurate with those of the node dynamics. In these situations, the study of the time scale competition between local dynamics and network variation is thus evidently important, which has yet escaped from attention and is the focus of this Rapid Communication.

In this work, we consider the synchronization problem in on-off complex networks with the on-off time scale being varied from a very small to a very large value compared to that of the associated coupled node dynamics. In the study, we find that when the on-off time scale is very small, the synchronization stability can be predicted by the static time-average coupling. As the on-off time scale is increased to the order comparable to the node dynamics, several interesting features are identified, among which the most exciting finding is that one of the traditional bound for synchronization due to short-wavelength bifurcations (SWBs) [14] nearly disappears, showing great advantage for the large-scale network fast synchronization. As the on-off time scale comes to the large limit, the synchronous behavior of the network can be explicitly explained in terms of Lyapunov exponents (LEs).

We consider a network of N dynamics units that interact with each other through connections with on-off couplings. For the convenience of investigation, the on-off coupling is uniform for all existed connections, since in this case we can use master stability function (MSF) rigorously to access the stability of synchronous states [4]. Thus, the topology of the studied network is then altered between the original graph and the isolated ensemble.

Based on these assumptions, the dynamics of each node can be described in terms of the following equations:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon(t)\sum_{j=1}^{N} g_{ij}\mathbf{h}(\mathbf{x}_j), \quad i = 1,2,\ldots,N. \tag{1}$$

Here $\mathbf{x} \in \mathbb{R}^m$ is the m-dimensional vector describing the state of the node, $\mathbf{F}(\mathbf{x}): \mathbb{R}^m \to \mathbb{R}^m$ governs the local dynamics of the nodes, $\mathbf{h}(\mathbf{x}): \mathbb{R}^m \to \mathbb{R}^m$ is a vectorial output function, $\varepsilon(t)$ is the on-off coupling strength, and $g=(g_{ij})$ is the coupling matrix for the original network with zero row sum, i.e., $\sum_{j=1}^{N} g_{ij}=0, \forall i$.

To be specific $\varepsilon(t)$, we need to consider a concrete dynamical system associated to each node. In the following, without lack of generality, we choose Rössler oscillators, whose state dynamics is described by $\dot{x}=-y-z$; $\dot{y}=x+ay$; $\dot{z}=b+z(x-c)$ with $\mathbf{x}=(x,y,z)$. Parameters are used for chaotic behavior $a=b=0.2$, $c=7$. To evaluate the time scale for the node dynamics, we calculated the average interpeak interval and obtained the typical dynamical time which is of order 1 s (in this case $T_{\text{typical}} \simeq 5.89$ s for x component), which will be considered when we choose the on-off period T. Specifically speaking, for the duration $nT < t < (n+\theta)T$, the network is switched on and $\varepsilon(t)=\varepsilon$; for the other time $(n+\theta)T < t < (n+1)T$, $n=0,1,2,\ldots$, the network is switched off and $\varepsilon(t)=0$. Here,

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θ is called the on-off rate in the range of 0–1, while $\theta=1$, the study is then recovered to the continuous case.

In this situation, the synchronous state is an invariant manifold, i.e., $\mathbf{x}_i=\mathbf{x}_s$, $\forall\, i$ with $\dot{x}_s=\mathbf{F}(\mathbf{x}_s)$. Now let us focus on the stability of the synchronized state, we perturb the state of each node around the synchronous states $\mathbf{x}_i(t)=\mathbf{x}(t)+\Delta\mathbf{x}_i(t)$ and expand Eq. (1) in the Taylor series of first order. Then the deviations $\Delta\mathbf{x}_i(t)=\mathbf{x}_i(t)-\mathbf{x}(t)$ satisfy the equations $\Delta\dot{\mathbf{x}}_i=\mathbf{JF}(\mathbf{x}_s)\Delta\mathbf{x}_i-\varepsilon(t)\Sigma_j g_{ij}\mathbf{Jh}(\mathbf{x}_s)\Delta\mathbf{x}_j$, where $\mathbf{J}$ denotes the Jacobian operator. Following the procedure of Pecora and Carroll [4], the linear stability of the synchronous state is determined by the above variational equations, which can be diagonalized into N blocks of the form,

$$\dot{\mathbf{y}}_k=[\mathbf{JF}(\mathbf{x}_s)-\varepsilon(t)\lambda_k\mathbf{Jh}(\mathbf{x}_s)]\mathbf{y}_k, \qquad (2)$$

where $\mathbf{y}_k$ represents different modes of perturbation from the synchronous state, and λ_k's are the eigenvalues of G, $k=1,\ldots,N$. To study synchronization properties with respect to different topologies, the variational equation should be computed as a function of a generic (complex) eigenvalue $\alpha+i\beta$. This leads to the definition of the following equation:

$$\dot{\mathbf{y}}=[\mathbf{JF}(\mathbf{x}_s)-\bar{\varepsilon}(t)(\alpha+i\beta)\mathbf{Jh}(\mathbf{x}_s)]\mathbf{y}, \qquad (3)$$

where $\bar{\varepsilon}(t)=\varepsilon(t)/\varepsilon$ is the normalized on-off coupling and we call the above equation as the coupling-dependent master stability equation (CMSE). The MSF then can be obtained by studying the largest Lyapunov exponent of the CMSE as a function of α and β, i.e., $\Lambda_{\max}(\alpha+i\beta)=\lim_{t\to\infty}\frac{1}{t}\ln|\mathbf{y}(t)/\mathbf{y}(0)|$. The last piece in the analysis for a given network is to study the sign of the $\Lambda_{\max}$ at the points $\alpha+i\beta=\varepsilon\gamma_k$ for the transverse modes $k=2,\ldots,N$. Only when all the transverse modes located in the negative region of the MSF, the synchronous state then becomes stable.

We first study the behavior of the system under the constraint of the small on-off time scale. Here, we choose

$$\mathbf{Jh}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

for the inner coupling. For the sake of simplicity, we consider the MSF as a function of α and on-off rate θ, omitting β. This is allowed when G has only real eigenvalues. Figure 1(a) reports the result for $T=0.1$ s, which is much more smaller than the typical dynamical time of nodes and can be considered as fast switching cases. We see that the stable region of the synchronous state is bounded by two hyperboliclike critical lines. The behavior can be well understood according to previous studies [6] by the spirit of the time average of the coupling matrix $G(t)$, which can be stated that if the network of oscillators synchronizes for the static time average of the topology defined as $\bar{G}=\frac{1}{T}\int_t^{t+T}G(\tau)d\tau$ then the network will synchronize with time-varying topology $G(t)$ if the time average is achieved sufficiently fast. Similarly, we can take the on-off coupling into account of the time-average coupling for the MSF. For this type MSF for the x-coupled Rössler, the stable region is a two-threshold region, i.e., $\alpha_1<\alpha<\alpha_2$ for the continuous coupling. For on-off coupling, the time-average coupling defined by $\bar{\alpha}=\frac{1}{T}\int_t^{t+T}\bar{\varepsilon}(t)\alpha d\tau=\alpha\theta$ should also admit the above relation $\alpha_1<\bar{\alpha}<\alpha_2$. We verified the relationship by plotting the critical boundaries since they satisfy $\alpha_{jc}\theta_{jc}=\alpha_j, j=1,2$ and found that they were in good agreement [see Fig. 1(a)]. However, the situation is changed and becomes a little more complicated that we cannot understand from the viewpoint of time-average coupling any more when the time scale for the on-off coupling increases, as we will see later.

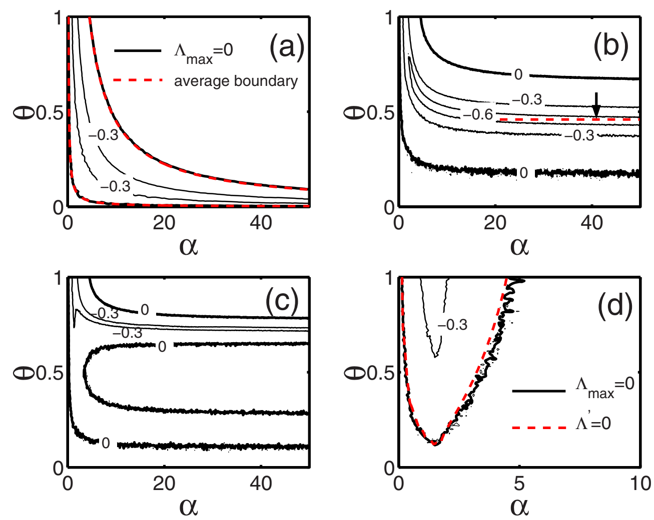

FIG. 1. (Color online) Contour plot of master stability function for x on-off coupled Rössler oscillators as a function of α and on-off rate θ for different on-off period T. (a) $T=0.1$ s for small time scale, (b) $T=3$ s, and (c) $T=6$ s for intermediate time scale; (d) $T=60$ s for large time scale. The red dashed lines in (a) and (d) are fitted results. Note two thresholds (α_1,α_2) for synchronization stability of the continuous coupling (corresponding to $\theta=1$) are about (0.14 and 4.48).

Now we come to the intermediate time scale for the on-off coupling that is the same order as the time scale of the node dynamics. As the time scale for the on-off coupling increases step by step from $T=0.1$ s, we observed that the hyperbolic boundaries no longer hold and gradually become parallel to each other for large α, and the stable region is enlarged much more than before. Figure 1(b) reports the typical case for $T=3$ s. As the time scale of the on-off coupling increases further, from about $T=4$ s, another stable region emerges that is also parallel to the abscissa [see Fig. 1(c)].

The feature we are primarily interested in under the intermediate time scale, as we can see from Figs. 1(b) and 1(c), is that the traditional upper bound nearly disappears for a considerable range of θ, which indicates that there is a transition from type III to type II MSF [2]. Type II MSF means a single threshold being expected and no size limits for large network synchronization, while for type III two bounds exist to be obstacles in synchronizing large-scale networks and much attempt hitherto have been made to enhance network synchronizability (e.g., review in [15]). Another important issue in network synchronization is the synchronization speed v [16,17], i.e., how fast can networked oscillators achieve synchrony to an expected precision. The speed is defined in terms of the synchronization error of networks Δ_{sync} by

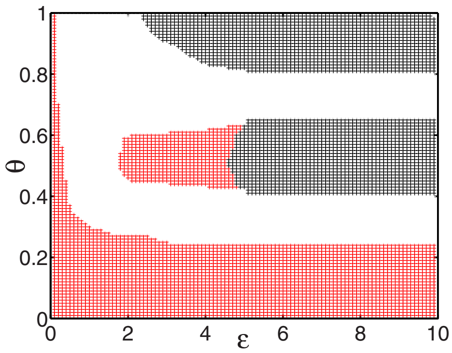

FIG. 2. (Color online) Plot of experimental results for two bidirectional on-off coupled Rössler circuits showing different behaviors in the parameter space. The blank, red (gray), and black regions correspond to synchronous, asynchronous, and blowout behaviors, respectively. Due to the experimental limitation of the apparatus, the coupling strength is limited up to $\varepsilon=10$ [corresponding to $\alpha=20$ in Fig. 1(c)].

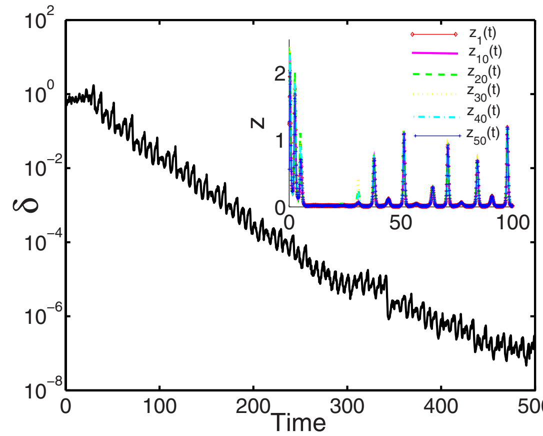

FIG. 3. (Color online) The plot shows the time evolution of the synchronization error $\delta(t)$ exponentially decays and the inset illustrates time series for some populations of the top predators for the parameter $T=1$, $\theta=0.1$, $\varepsilon=100$. Here, year can be regarded as the time unit. The model exhibits a rhythm with a period of about ~6 year. The result means intensive interaction (coupling) among communities in animals' active season for just few months can synchronize very large spatially coupled zones.

$\Delta_{\mathrm{sync}}(t)=\Delta_{\mathrm{sync}}(0)e^{-vt}$. Our previous study showed that v is closely related to the largest Lyapunov exponent corresponding to the least stable transverse mode in Eq. (2); that is, $v \simeq -\max\{\Lambda_{\max}(\varepsilon\gamma_k)\}$, $k=2,\ldots,N$. The interesting phenomenon under this time scale is that the speed of the network synchronization can be accelerated a lot since for some θ all the transverse modes can be placed in the region corresponding large negative $\Lambda_{\max}$ [see, e.g., the red dashed line indicated by the arrow for $\theta=0.45$, $v=-\Lambda_{\max}\gtrsim 0.8$ in Fig. 1(b)] and thus converge to zero exponentially nearly at the same rate. In fact, in this situation the speed is almost the same for different networks independent of its structures as long as the coupling strength is strong enough.

To identify this interesting synchronous behavior on real physical systems, we conducted an electronic circuit composed of two coupled Rössler oscillators to examine the phenomena. The on-off coupling is produced by a square-wave generator and added to the system via bilateral switches. The circuit runs in the realm of a few kHz. Specifically, both time scales for the oscillators and for the on-off coupling are about $1/2500=4\times10^{-4}$ (s), corresponding to the case of Fig. 1(c). Figure 2 illustrates some typical regions of the circuit, from which we can observe that the synchronous region is roughly the same to that of the stability analysis [see Fig. 1(c)]. Besides, the asynchronous and blowout behaviors are also identified. Notice that due to the disparity of elements of the two oscillators, we say that synchronization happens if the difference between the variable voltages tends to a small range around zero. More details of the experiment will be given elsewhere.

In fact, as the time scale of on-off coupling continually increases, more stable regions appear parallel to the abscissa, and the width of these regions inevitably becomes thinner and thinner. Finally, as the time scale is large enough (about 20 s in this example), all the long parallel stable regions become too thin to be observed and vanish at last, converging to another region [Fig. 1(d) for $T=60$]. The behavior under large time scale of on-off coupling actually can be explained in terms of LEs. From the physical definition of LE, the difference $\delta X(0)$ of the two nearby trajectories will diverge (or converge) as $e^{\Lambda t}$. During period θT the coupling is switched on, the difference evolves as $e^{\Lambda_{\perp}t}$, where $\Lambda_{\perp}$ is the largest transverse LE for the continuous coupling. In the next step as the coupling is switched off, the difference evolves as $e^{\Lambda^0_{\max}t}$ for period $(1-\theta)T$, where the $\Lambda^0_{\max}$ is the largest LE for sole oscillator since the coupling is in absence. The process repeats again and again, and the resulting average difference then can be expressed by $\frac{\delta X(t)}{\delta X(0)} \sim e^{[\Lambda_{\perp}\theta+\Lambda^0_{\max}(1-\theta)]t}$; that is, the average LE for the large time scale

$$\Lambda' = \Lambda_{\perp}\theta + \Lambda^0_{\max}(1-\theta). \qquad (4)$$

Whether stable synchronization can be obtained or not is completely determined by the competition of duration of the two processes. We verified the explanation by plotting the critical boundary by letting $\Lambda'=0$ and found that they were in good agreement. It is worth noting that the argument seems universal for this kind of coupling scheme regardless of the choice of T, but our results show that it works only at the large time scale. This is due to the fact that the LE is the long-time average rate of divergence (or convergence) of two neighboring trajectories in the phase space, and it succeeds only in those cases when the dynamics process is long enough to ensure its working context.

As an application of the problem studied above, we consider an ecological system, the foodweb model [18], with some practical considerations. The individual dynamics is ruled by $\mathbf{F}(\mathbf{x})=[x-\frac{0.2xy}{1+0.05x},-y+\frac{0.2xy}{1+0.05x}-yz,-10(z-0.006)+yz]^T$, with

$$\mathbf{Jh} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The model describes a standard three level vertical food chain, where the vegetation x is consumed by herbivores y, which in turn are preyed on by the top predators z. The model was motivated by the fact that in many ecological systems, the frequency of a population remains relatively constant (the "4- and 10-year cycle" of wildlife are often found) over time although there are erratic changes in abundance, such as the most celebrated Canadian hare-lynx cycle, which follow a tight rhythm with a period of $\sim$10 years [19]. Here, we illustrate our argument by direct simulation of a nearest-neighbor-coupled ring with on-off coupling, in which the on-off coupling can be accounted for the seasonality. For example, many species (such as guns, zebras, and flamingoes in Africa) migrate for food or mating in the dry season, which can be regarded as strong interaction. In contrast, some animals (such as some bears and snakes) may hibernate as winter comes and thus the interaction among communities ceases until the next spring. It is important to notice that in this case, both the time scales for the population dynamics and for the on-off coupling are of the same order (1 year). We defined the synchronization error $\delta(t)=\frac{1}{N}\sum_{i=1}^{N}(|x_i-\bar{x}|+|y_i-\bar{y}|+|z_i-\bar{z}|)$ with $\bar{x}=\frac{1}{N}\sum_{i=1}^{N}x_i$, $\bar{y}=\frac{1}{N}\sum_{i=1}^{N}y_i$, and $\bar{z}=\frac{1}{N}\sum_{i=1}^{N}z_i$. Figure 3 reports the result for 50 coupled patches for $T=1$ year, $\theta=0.1$, from which we can see that all communities evolve into a common synchronous state exponentially in the course of time. It should be noted that if we try to synchronize the above system with the traditional continuous coupling (i.e., $\theta=1$), there will be a size upper limit ($N_{\max}\simeq 14$ in this example) [14], above which no synchronous state can be obtained whatever manipulating the coupling strength.

In conclusion, we have systematically studied the synchronous behaviors of on-off complex networks, showing that the role of the time scale for network variation is of critical importance for network dynamics. Especially when the time scale of on-off coupling is comparable to that of associated node dynamics, the stable region is dramatically enlarged and the speed of synchronization is accelerated considerably. Although the study is a simplification for more general time-varying cases, our results indicate that among different time scales of network variation for network synchronization, the intermediate time scale may exhibit great advantages over other time scales. We believe our research will provide fresh insight into many collective phenomena in nature and also hints for engineering design.

It is a pleasure to thank Dr. G. X. Qi for technical comments on the manuscript.

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